

# On the energy-minimizing steady states of a thin film equation

Almut Burchard  
University of Toronto  
almut@math.utoronto.ca

Marina Chugunova  
University of Toronto  
chugunom@math.utoronto.ca

Benjamin K. Stephens  
University of Washington  
benstph@math.washington.edu

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## Abstract

Steady states of the thin film equation  $u_t + [u^3(u_{\theta\theta\theta} + \alpha^2 u_\theta - \sin \theta)]_\theta = 0$  are considered on the periodic domain  $\Omega = (-\pi, \pi)$ . The equation defines a generalized gradient flow for an energy functional that controls the  $H^1(\Omega)$ -norm. The main result establishes that there exists for each given mass a unique nonnegative function of minimal energy. This minimizer is symmetric decreasing about  $\theta = 0$ . For  $\alpha < 1$  there is a critical value for the mass at which the minimizer has a touchdown zero. If the mass exceeds this value, the minimizer is strictly positive. Otherwise, it is supported on a proper subinterval of  $\Omega$  and meets the dry region at zero contact angle. A second result explores the relation between strict positivity and exponential convergence for steady states. It is shown that positive minimizers are locally exponentially attractive, while the distance from a steady state with a dry region cannot decay faster than a power law.

# 1 Introduction

Degenerate fourth order parabolic equations of the form

$$u_t + \nabla \cdot (u^n \nabla \Delta u) + \text{lower order terms} = 0$$

are commonly used to model the evolution of thin liquid films on the surface of a solid. Here,  $u(x, t)$  describes the thickness of the fluid at time  $t$  at the point  $x$ , the fourth derivative term models surface tension, and the exponent  $n > 0$  is determined by the boundary condition between the liquid and the surface of the cylinder. A particularly interesting case is  $n = 3$ , which corresponds to a “no-slip” boundary condition.

In this paper, we study the equation

$$u_t + [u^n (u_{\theta\theta\theta} + \alpha^2 u_{\theta} - \sin \theta) + \omega u]_{\theta} = 0, \quad \theta \in \Omega = (-\pi, \pi) \quad (1.1)$$

with periodic boundary conditions. For  $n = 3$  and  $\alpha = 1$ , this describes the evolution of a thin liquid film on the outside of a horizontal cylinder that rotates slowly about its axis, see Figure 1. The film is assumed to be uniform along the axis of the cylinder, and its thickness at time  $t$  and angle  $\theta$  (measured from the bottom) is given by the function  $u(\theta, t)$ . In Eq. (1.1), the first summand in the parentheses models surface tension, the next term is a correction due to the curvature of the cylinder, and the third term models gravitational drainage. The last term models convection due to rotation. We have scaled the units of length and time so that the surface tension and gravitational terms appear with coefficient one; the coefficient  $\alpha \geq 0$  is a geometric constant, and  $\omega$  is proportional to the speed of rotation. Here, we will study the non-rotating cylinder with  $\omega = 0$ . We are mainly interested in the case where  $n = 3$  and  $\alpha = 1$ , but find it illuminating to also consider other values of  $n$  and  $\alpha$ .

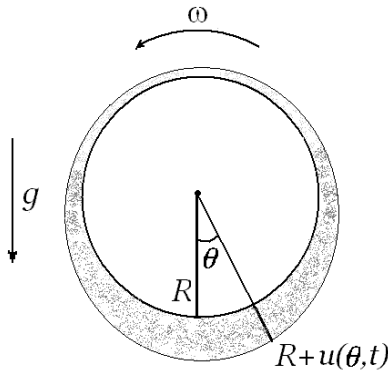


Figure 1: Liquid film on the outer surface of a rotating horizontal cylinder in the presence of gravity.

The model in Eq. (1.1) (with  $n = 3$  and  $\alpha = 1$ ) was first derived and studied by Pukhnachov [24, 25]. Note that Pukhnachov uses slightly different coordinates (with  $\theta = \frac{3\pi}{2}$  at the bottom of the cylinder), and a different scaling (where  $\omega = 1$ ). The derivation relies on the lubrication approximation; it assumes that the liquid film is very thin compared to the radius of the cylinder, and that the rotation is slow enough to neglect centrifugal forces. Pukhnachov’s model refines an earlier model of Moffatt [21] by including surface tension. The second order curvature term in the equation is reminiscent of a porous-medium equation, but appears with the opposite sign, resulting in a long-wave

instability [6]. Interesting numerical and asymptotical analysis along with numerous open questions can be found in [1, 2, 14].

A fundamental technical problem in thin film equations is well-posedness, i.e., to show that non-negative initial data give rise to unique nonnegative solutions that depend continuously on the data. The difficulty is that solutions of fourth order parabolic equations generally do not satisfy maximum or comparison principles, and linearization leads to semigroups that do not preserve positivity. To give a simple example, the function  $u(x, t) = 1 + t \cos(x)$ , which solves Eq. (1.1) in the linear case  $n = 0$  with  $\alpha = 1$  and  $\omega = 0$ , ceases to be positive for  $t > 1$ . Thus, positivity of solutions of thin film equations is a nonlinear phenomenon. Other relevant problems in the area concern long-term behavior, finite-time blow-up, and the interface between wet ( $u > 0$ ) and dry ( $u = 0$ ) regions. These problems have been studied rigorously in a vast body of papers since the pioneering article of Bernis and Friedman [4], see for example [3, 5, 6, 10, 22] and references therein. There is an even larger literature that studies the properties of physically relevant solutions through asymptotic expansions, numerical analysis, and laboratory experiments.

Bernis and Friedman proved that initial-value problems for thin film equations can be solved in suitable classes of nonnegative weak solutions. An important technical contribution was their use of an *entropy functional* that decreases with time along solutions. A few years later, the so-called  $\alpha$ -entropies were independently discovered by Bertozzi and Pugh [5] and by Beretta, Bertsch, and dal Passo [3]. Since then, other families of entropies have been found [8, 13, 16]. Entropy functionals are the basis for results on short- and long-term existence, positivity, finite speed of propagation, regularity, blow-up, and the long-time behavior of solutions.

For  $0 < n \leq 1$ , well-posedness and convergence to steady states have recently been established by treating Eq. (1.1) as a gradient flow on a space of measures endowed with the Wasserstein distance [20, 22], where the exponent  $n$  appears as a mobility parameter [9]. These gradient flow techniques also take advantage of energy and entropy dissipation. However, Wasserstein distances with mobility  $n > 1$  are not well understood, and well-posedness remains an open problem.

By a *solution* of Eq. (1.1) we mean a nonnegative function  $u \in L^2((0, T), H^2(\Omega))$  that satisfies

$$\int_0^T \int_{\Omega} \left\{ u \phi_t - (u_{\theta\theta} + \alpha^2 u + \cos \theta)(u^n \phi_{\theta})_{\theta} + \omega u \phi_{\theta} \right\} d\theta dt = 0$$

for every smooth test function with compact support in  $\Omega \times (0, T)$ . This agrees with the strongest notion of generalized solutions from [4, 5]. For a class of equations that includes Eq. (1.1) with  $n = 3$ , long-time existence of such solutions was recently proved in [10]. These solutions are widely believed to be unique.

Questions about steady states appear in many applications. When do steady states exist, when are they uniquely determined by their mass, are they strictly positive or do they exhibit dry regions, and under what conditions are they stable? Do steady states attract all bounded solutions? When can we expect exponential convergence? Linearizations of Eq. (1.1) about steady states were examined analytically and numerically in [11, 12].

For  $\omega = 0$ , Eq. (1.1) defines a generalized gradient flow for the *energy*

$$E(u) = \frac{1}{2} \int_{\Omega} u_{\theta}^2 - \alpha^2 u^2 d\theta - \int_{\Omega} u \cos \theta d\theta, \quad (1.2)$$

in the sense that  $u_t = [u^n (\frac{\delta E}{\delta u})_{\theta}]_{\theta}$ . Here,  $\frac{\delta E}{\delta u}$  denotes the  $L^2$ -gradient of  $E$ . This implies the dissipation

estimate

$$\frac{d}{dt}E(u(\cdot, t)) = - \int_{\Omega} u^n \left( \frac{\delta E}{\delta u} \right)_{\theta}^2 d\theta \leq 0. \quad (1.3)$$

The subject of this paper are the minimizers of  $E$  on the set of nonnegative  $2\pi$ -periodic functions of a given mass

$$C_M = \left\{ u \in H^1(\Omega) \mid u \geq 0, \int_{\Omega} u d\theta = M \right\}$$

and their role in the dynamics. Note that for  $\alpha = 1$ ,  $E$  is convex but not strictly convex. For  $\alpha < 1$ , the functional is strictly convex, and has a unique critical point on  $C_M$ , which is a global minimizer. For  $\alpha > 1$ , it is not convex, and we may expect multiple critical points.

We will show that  $E$  has a unique minimizer on  $C_M$  for each value of  $\alpha$  and each mass  $M > 0$ , see Theorem 1. These minimizers may have dry regions; in that case, the contact angle of the fluid film is zero, see the Figure 2. In particular, our result establishes the existence of zero contact angle steady states for Eq (1.1) if  $M(1 - \alpha^2) \leq 2\pi$ . Our proof relies on symmetric decreasing rearrangements and the first variation of the energy.

The minimizers are time-independent solutions of Eq. (1.1) with  $\omega = 0$ . Additional steady states may arise for  $\alpha > 1$  as saddle points of the energy. For any value of  $\alpha$  and  $M$ , there is also a continuum of steady states with have non-zero contact angles, analogous to the steady states in [15], whose role in the evolution remains unclear.

We expect that for  $\alpha \leq 1$ , the unique energy minimizer should attract all solutions on  $C_M$  as  $t \rightarrow \infty$ . Unfortunately, in the absence of a proper well-posedness theory, Lyapunov's theorem is not sufficient to support this expectation. In Theorem 2 we provide partial results in that direction. If the energy-minimizing steady state is strictly positive, then it exponentially attracts all solutions in a neighborhood. On the other hand, for  $n > \frac{3}{2}$ , a steady state that has a dry interval of positive length cannot be exponentially attractive. In particular for  $n = 3$  and  $\alpha = 1$ , the distance between the solution and the minimizer decays no faster than  $t^{-\frac{2}{3}}$ . Our proof combines energy and entropy inequalities in the spirit of [3, 5, 7, 26].

All our results are easily adapted to the long-wave stable case where the sign of the second order term is reversed. In that case, the energy-minimizing steady state is strictly positive and locally exponentially attractive so long as  $M(1 + \alpha^2) > 2\pi$ . For  $M(1 + \alpha^2) < 2\pi$ , the energy minimizer has a dry interval of positive length, which it meets at zero contact angles. This contrasts with a theorem of Laugesen and Pugh that excludes zero contact angle steady states for the corresponding thin film equation without the sine term [15].

## 2 Identification of energy minimizers

We begin by showing that for every  $M > 0$  there exists a function  $u$  with mass  $M$  that minimizes the energy. The first lemma provides the necessary global bounds on the functional.

**Lemma 1 (Lower bound on the energy.)**  *$E$  is bounded from below and coercive on  $C_M$ .*

*Proof.* Using that  $u$  is nonnegative and has mean  $\frac{M}{2\pi}$ , we estimate

$$\int_{\Omega} u^2 dx \leq M \|u\|_{L^\infty}, \quad \|u\|_{\infty} \leq \frac{M}{2\pi} + \sqrt{\pi} \|u_{\theta}\|_{L^2}.$$

It follows that

$$E(u) \geq \frac{\pi}{2} \left( \|u\|_\infty - \frac{M}{2\pi}(1 + \alpha^2) \right)^2 - \frac{M^2}{4\pi} \alpha^2 (2 + \alpha^2) - M. \quad (2.1)$$

which is clearly bounded below. A similar estimate shows that  $E$  grows quadratically as  $\|u\|_{H^1} \rightarrow \infty$ .  $\square$

The lemma implies that minimizing sequences are bounded in  $H^1$ . Passing to a subsequence, we can construct a minimizing sequence  $\{u_j\}_{j \geq 1}$  that converges weakly in  $H^1$  and strongly in  $L^2$  to some function  $u$  in  $C_M$ . Since  $E$  is weakly lower semicontinuous on  $H^1$ ,  $u$  is the desired minimizer. We next describe some properties of the minimizers.

**Lemma 2 (Symmetry.)** *Minimizers of  $E$  on  $C_M$  are symmetric decreasing about  $\theta = 0$ .*

*Proof.* For  $u \in C_M$ , let  $u^\#$  be the unique symmetric decreasing function of  $\theta$  that is equimeasurable to  $u$ . Classical results about symmetric decreasing rearrangement ensure that  $u^\# \in C_M$ , and that

$$\|u^\#\|_{L^2} = \|u\|_{L^2}, \quad \|u_\theta^\#\|_{L^2} \leq \|u_\theta\|_{L^2}, \quad \int_\Omega u^\# \cos \theta \, d\theta \geq \int_\Omega u \cos \theta \, d\theta, \quad (2.2)$$

see [23]. It follows that

$$E(u^\#) \leq E(u).$$

If  $u$  is a minimizer, then  $E(u^\#) = E(u)$ , and in particular, the third inequality in Eq. (2.2) must hold with equality. Since the cosine is *strictly* symmetric decreasing, this forces  $u$  to be symmetric decreasing as well [17, Theorem 3.4].  $\square$

The Euler-Lagrange equation for the minimizer is given by

$$u_{\theta\theta} + \alpha^2 u + \cos \theta = \lambda \quad \text{on } \{\theta \in \Omega \mid u(\theta) > 0\}, \quad (2.3)$$

where  $\lambda$  is a Lagrange multiplier associated with the mass constraint. By considering the first variation of  $E$  with respect to nonnegative functions that need not vanish outside the support of  $u$ , we see that

$$u_{\theta\theta} + \alpha^2 u + \cos \theta \leq \lambda \quad \text{on } \Omega$$

as a distribution. This suggests that the first derivative of a minimizer should vanish at the boundary of its support, i.e., the film meets the surface of the cylinder at zero contact angle. The next lemma confirms this suspicion.

**Lemma 3 (Zero contact angle.)** *Let  $u$  be a minimizer of  $E$  on  $C_M$ . If  $u$  has its first positive zero at  $\theta = \tau$ , then  $u_\theta(\tau_-) = 0$  and  $u_{\theta\theta}(\tau_-) > 0$ . In particular,  $u \in \mathcal{C}^{1,1}(\Omega)$ .*

*Proof.* By Eq. (2.3),  $u$  has one-sided derivatives of arbitrary order at  $\tau$ , and  $u_\theta(\tau_-) \leq 0$ . Suppose that  $u_\theta(\tau_-) < 0$ . We will modify  $u$  to construct a new valid competitor with lower energy. If  $\tau < \pi$ , set  $v(\theta) = u(\phi(\theta))$ , where  $\phi : \Omega \rightarrow \Omega$  is the bi-Lipschitz map defined by

$$\phi(\theta) = \begin{cases} \frac{1}{2}(\theta + \tau - \varepsilon), & \tau - \varepsilon \leq \theta \leq \tau + \varepsilon, \\ 2\theta - \tau - 2\varepsilon, & \tau + \varepsilon \leq \theta \leq \tau + 2\varepsilon, \\ \theta, & \text{otherwise.} \end{cases}$$

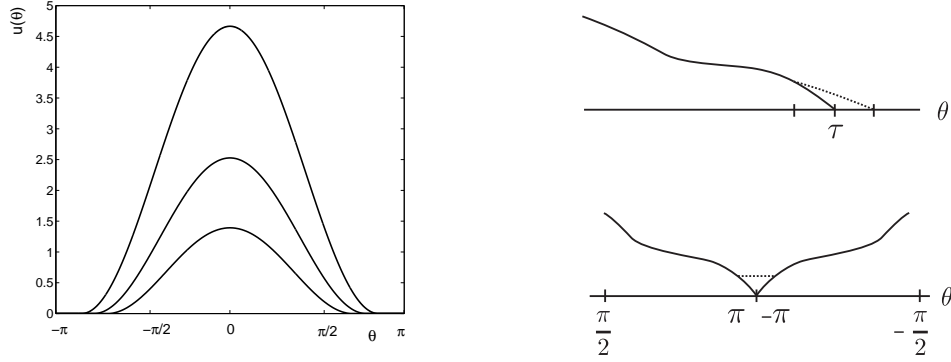


Figure 2: On the left: Steady states for  $\alpha = 1$  and initial data  $u_0 = 0.5, 1, 2$ . As the mass is getting smaller, the minimizer becomes more concentrated, and as mass goes to infinity the support tends to  $[-\pi, \pi]$ . On the right: Energy-decreasing variations resulting from non-zero contact angles.

Choose  $\varepsilon > 0$  small enough so that  $u$  vanishes on  $[\tau, \tau + 2\varepsilon]$ . Then  $v$  vanishes on  $[\tau + \varepsilon, \tau + 2\varepsilon]$ . The difference between the leading terms in the energy integrals is given by

$$\frac{1}{2} \int_{\Omega} v_{\theta}^2 d\theta - \frac{1}{2} \int_{\Omega} u_{\theta}^2 d\theta = \frac{1}{2} \int_{\tau-\varepsilon}^{\tau+\varepsilon} v_{\theta}^2 d\theta - \frac{1}{2} \int_{\tau-\varepsilon}^{\tau} u_{\theta}^2 d\theta = -\frac{\varepsilon}{4} u_{\theta}(\tau_{-})^2 + O(\varepsilon^2),$$

see Figure 2, top right. Since the remaining terms contribute only corrections of order  $O(\varepsilon)^2$  to the energy difference, it follows that

$$E(v) - E(u) = -\frac{\varepsilon}{4} u_{\theta}(\tau_{-})^2 + O(\varepsilon)^2.$$

If  $\tau = \pi$ , the same estimate holds (by the symmetry of  $u$ ) for  $v(\theta) = u(\min\{|\theta|, \pi - \frac{1}{4}\varepsilon\})$ , see Figure 2, bottom right. In either case, the mass of  $v$  is  $M' = \int_{\Omega} v d\theta = M + O(\varepsilon)^2$ . We finally set  $w(\theta) = \frac{M}{M'} v(\theta)$ , which has the correct mass and satisfies  $E(w) = E(v) + O(\varepsilon^2) < E(u)$  for  $\varepsilon$  sufficiently small, a contradiction. We conclude that  $u_{\theta}(\tau_{-}) = 0$ , proving the first claim.

To prove the second claim, we analyze the sign of the first non-vanishing left derivative of  $u$  at  $\tau$ . Clearly,  $u_{\theta\theta}(\tau_{-}) \geq 0$ . Suppose that  $u_{\theta\theta}(\tau_{-}) = 0$ . Differentiating Eq. (2.3), we obtain  $u_{\theta\theta\theta}(\tau_{-}) = \sin \tau \geq 0$ . Since  $\tau$  is the first positive zero of  $u$ , the derivative  $u$  from the left cannot be positive, and so  $\theta = \pi$  is the only possibility. Differentiating once more, we obtain  $u_{\theta\theta\theta}(\pi_{-}) = -1$ , which is the wrong sign for  $u$  to have a minimum at  $\pi$ . It follows that  $u_{\theta\theta}(\tau_{-}) > 0$ , as claimed.  $\square$

We are now ready to compute the minimizers of  $E$  on  $C_M$  explicitly. A particular solution of the Euler-Lagrange equation in Eq. (2.3) is given by

$$u^0(\theta) = \begin{cases} -\frac{1}{2}\theta \sin \theta, & \alpha = 1, \\ \frac{1}{1-\alpha^2} \left( \cos \theta - \frac{1+\alpha^2}{2\alpha} \cos(\alpha\theta) \right), & \alpha \neq 1, \end{cases} \quad (2.4)$$

and the general solution can be represented as

$$u(\theta) = A \cos(\alpha\theta) + B \sin(\alpha\theta) + \frac{\lambda}{\alpha^2} + u^0(\theta). \quad (2.5)$$

The following theorem summarizes our results. The statement is illustrated in Figure 3.

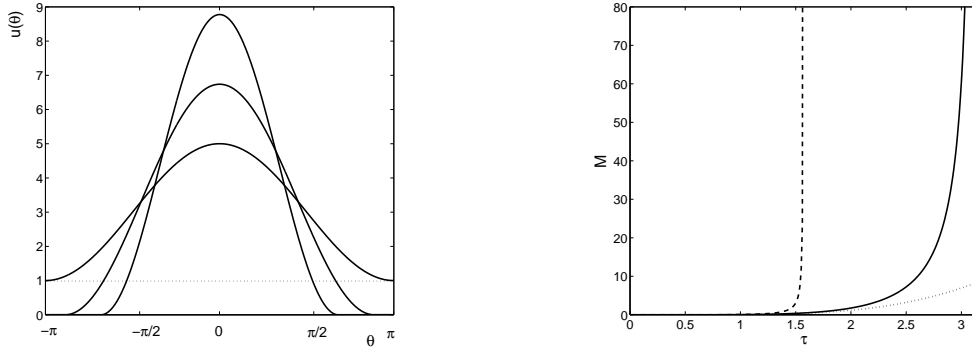


Figure 3: On the left: Numerical steady states for  $\alpha = 0.5, 1, 2$  with initial data  $u_0 = 3$ . On the right: Mass versus half length of the compact support for  $\alpha = 2, 1, 0.5$ .

**Theorem 1 (Description of the energy minimizers.)** *Let  $E$  be the energy functional in Eq. (1.2). For each  $M > 0$ ,  $E$  has a unique nonnegative minimizer of mass  $M$ . The minimizer is strictly symmetric decreasing on its support. It is of class  $C^{1,1}$  and depends continuously on  $M$  in  $C^{1,1}$ . It increases with  $M$  in the sense that for any pair of minimizers  $u_1, u_2$  of mass  $M_1, M_2$ ,*

$$M_1 < M_2 \implies u_1(\theta) < u_2(\theta), \quad \theta \in \text{support}(u_1).$$

- If  $M(1 - \alpha^2) > 2\pi$ , the minimizer is strictly positive and given by

$$u(\theta) = \frac{M}{2\pi} + \frac{1}{1 - \alpha^2} \cos \theta; \quad (2.6)$$

- if  $M(1 - \alpha^2) < 2\pi$ , the minimizer is given by

$$u(\theta) = A(\tau)(\cos(\alpha\theta) - \cos(\alpha\tau)) + u^0(\theta) - u^0(\tau), \quad |\theta| \leq \tau \quad (2.7)$$

for some  $\tau$  with  $\max\{\alpha, 1\}\tau < \pi$ , and vanishes for  $|\theta| \geq \tau$ . The coefficient is determined by  $A(\tau) = \frac{u_\theta^0(\tau)}{\alpha \sin(\alpha\tau)}$ , where  $u^0$  is the special solution from Eq. (2.4);

- if  $M(1 - \alpha^2) = 2\pi$ , then Eq. (2.7) for  $\tau = \pi$  coincides with Eq. (2.6), and  $u(\theta) = \frac{1 + \cos \theta}{1 - \alpha^2}$ .

*Proof.* Fix  $M > 0$ . By Lemma 1, there exists a minimizer  $u$  of mass  $M$ , and by Lemma 2, it is symmetric decreasing about  $\theta = 0$ . If the positivity constraint is not active, then Eq. (2.5) holds for all  $\theta \in \Omega$ . Since the minimizer is smooth, periodic, and has mass  $M$ , we conclude that  $\alpha < 1$  and Eq. (2.6) holds. In order for  $u$  to be nonnegative and symmetric decreasing we must have  $M(1 - \alpha^2) \geq 2\pi$ . In that region,  $u$  is clearly strictly increasing in  $\theta$ .

If, on the other hand, the positivity constraint is active, then the minimizer  $u$  is positive on some interval  $(-\tau, \tau)$  and vanishes for  $|\theta| \geq \tau$ . By Lemma 3,  $u \in C^{1,1}(\Omega)$  and  $u_\theta(\pm\tau) = 0$ . On its support,  $u$  is given by Eq. (2.5). Since  $u$  and  $u^0$  are even,  $B = 0$ . The Dirichlet condition at  $\tau$  allows to eliminate

$\lambda$ , the Neumann condition determines  $A$ , and we find that Eq. (2.7) holds. If  $\max\{\alpha, 1\}\tau < \pi$ , we claim that  $u$  is indeed nonnegative, symmetric decreasing in  $\theta$ , and strictly increasing with  $\tau$ . To see this, we differentiate Eq. (2.7), and use Lemma 3 to obtain

$$\frac{dA}{d\tau} \cdot \alpha \sin \alpha\tau = -u_{\theta\tau}(\tau; \tau) = u_{\theta\theta}(\tau; \tau) > 0.$$

By the chain rule, and using once more that  $u_\theta(\tau; \tau) = 0$ , we have

$$u_\tau(\theta; \tau) = \frac{dA}{d\tau} \cdot (\cos(\alpha\theta) - \cos(\alpha\tau)) > 0 \quad \text{for } |\theta| < \tau.$$

Since  $u$  vanishes identically when  $M = 0$ , this confirms that  $u$  is positive and strictly symmetric decreasing for  $|\theta| < \tau$ . We use that  $u_\theta(\tau; \tau) = 0$  to compute

$$\frac{dM}{d\tau} = \frac{dA}{d\tau} \int_{-\tau}^{\tau} \cos(\alpha\theta) - \cos(\alpha\tau) d\theta > 0,$$

and infer that we can solve for  $\tau = \tau(M)$  as a strictly increasing smooth function of  $M$ . By the chain rule and the inverse function theorem,

$$\frac{d}{dM} u(\theta; \tau(M)) = \frac{\cos(\alpha\theta) - \cos(\alpha\tau)}{\int_{-\tau}^{\tau} \cos(\alpha\theta') - \cos(\alpha\tau) d\theta'} > 0,$$

establishing the desired continuity and monotonicity of  $u$  with respect to  $M$  in the range where Eq. (2.7) is valid and  $\max\{\alpha, 1\}\tau < \pi$ .

We need to determine the ranges where Eq. (2.6) and (2.7) hold. For  $\alpha < 1$ ,  $E$  is strictly convex. If  $M \geq \frac{2\pi}{1-\alpha^2}$ , the function defined by Eq. (2.6) is nonnegative and provides the unique minimizer of  $E$  on  $C_M$ . If  $M < \frac{2\pi}{1-\alpha^2}$ , the positivity constraint is active, and we compute from Eq. (2.7) that  $M \rightarrow 0$  as  $\tau \rightarrow 0$  and  $M \rightarrow \frac{2\pi}{1-\alpha^2}$  as  $\tau \rightarrow \pi_-$ . Continuous dependence on  $M$  follows, since Eq. (2.7) agrees with Eq (2.6) at  $M = \frac{2\pi}{1-\alpha^2}$ .

For  $\alpha = 1$ ,  $E$  is convex, but not strictly convex on  $C_M$ . The positivity constraint is active, because  $E$  is not bounded below without it; for instance,  $E\left(\frac{M}{2\pi} + t \cos \theta\right) = -\pi t$ . Note that we must have  $\tau < \pi$ , because the particular solution  $u^0(\theta) = -\frac{1}{2}\theta \sin \theta$  from Eq. (2.4) cannot be continued as a differentiable periodic function across  $\theta = \pi$ , in violation of Lemma 3. It is easy to check from Eq. (2.7) that  $M \rightarrow 0$  as  $\tau \rightarrow 0$ , and  $M \rightarrow \infty$  as  $\tau \rightarrow \pi$ .

For  $\alpha > 1$ , the energy is a non-convex quadratic function on  $C_M$ , and hence the positivity constraint is active and  $u$  is given by Eq. (2.5) on some interval  $(-\tau, \tau)$ . Upon closer inspection of Eq. (2.5), we see that  $\alpha\tau < 1$ , since otherwise  $u$  fails to be symmetric decreasing. Since  $M \rightarrow 0$  as  $\tau \rightarrow 0$  and  $M \rightarrow \infty$  as  $\tau \rightarrow \alpha^{-1}\pi$ , the theorem follows.  $\square$

### 3 Convergence to minimizers

In this section, we will prove a lower bound on the speed at which solutions of Eq. (1.1) with  $\omega = 0$  can converge to critical points on the boundary of the positive cone. We establish this bound for two classes of solutions: For strictly positive, classical solutions when  $n > \frac{3}{2}$ , and for the strong generalized solutions constructed for  $n = 3$  in [10, Section 3].



Our bound uses the *entropy method*, applied to the functional

$$S(u) = \int_{\Omega} u^{-\beta} d\theta, \quad (3.1)$$

where  $\beta = n - \frac{3}{2}$ . Strictly speaking,  $S$  is not an entropy for Eq. (1.1), because it may increase as well as decrease along solutions. One of the reasons is that the porous medium term  $\alpha^2(u^n u_{\theta})_{\theta}$  appears in Eq. (1.1) with the unfavorable sign. Still, the standard entropy methods yields a useful differential inequality for  $S$ .

**Lemma 4 (Entropy inequality for classical solutions)** *Fix  $n > \frac{3}{2}$  and let  $S$  be given by Eq. (3.1) with  $\beta = n - \frac{3}{2}$ . For every strictly positive classical solution  $u$  of Eq. (1.1), there exist constants  $S_0$  and  $K_0$  such that*

$$S(u(\cdot, t)) \leq S_0 + K_0 t. \quad (3.2)$$

*Proof.* Let  $S_0 = S(u(\cdot, 0))$  and  $E_0 = E(u(\cdot, 0))$  be the initial values of the entropy and energy, and set  $c_n = (n - \frac{3}{2})(n - \frac{1}{2})$ . We will show that

$$\frac{d}{dt} S(u(\cdot, t)) \leq K_0,$$

where

$$K_0 = c_n \left\{ \frac{M\alpha^2}{4} \left[ \frac{M}{2\pi} (1 + \alpha^2) + \left( \frac{2(E_0 + M)}{\pi} + \frac{M^2}{4\pi^2} \alpha^2 (2 + \alpha^2) \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} + 2\sqrt{M\pi} \right\}.$$

The claim then follows by integrating along the solution.

To see the differential inequality, we use the Eq. (1.1) and integrate by parts,

$$c_n^{-1} \frac{dS(u)}{dt} = \int_{\Omega} u^{-\frac{1}{2}} u_{\theta} u_{\theta\theta\theta} d\theta + \alpha^2 \int_{\Omega} u^{-\frac{1}{2}} u_{\theta}^2 d\theta - \int_{\Omega} u^{-\frac{1}{2}} u_{\theta} \sin \theta d\theta. \quad (3.3)$$

The first summand in Eq. (3.3) we integrate again by parts,

$$\int_{\Omega} u^{-\frac{1}{2}} u_{\theta} u_{\theta\theta\theta} d\theta = - \int_{\Omega} u^{-\frac{1}{2}} u_{\theta\theta}^2 d\theta + \frac{1}{2} \int_{\Omega} u^{-\frac{3}{2}} u_{\theta}^2 u_{\theta\theta} d\theta =: -A + \frac{1}{2}B,$$

and integrate by parts once more to see that  $B = \frac{1}{2} \int_{\Omega} u^{-\frac{5}{2}} u_{\theta}^4 d\theta$ . After collecting terms, the first summand becomes

$$-A + B - \frac{1}{2}B = - \int_{\Omega} u^{-\frac{1}{2}} \left( u_{\theta\theta} - \frac{1}{2} u^{-1} u_{\theta}^2 \right)^2 d\theta = -4 \int_{\Omega} u^{\frac{1}{2}} \left( (u^{\frac{1}{2}})_{\theta\theta} \right)^2 d\theta.$$

The second summand in Eq. (3.3) we also integrate by parts,

$$\alpha^2 \int_{\Omega} u^{-\frac{1}{2}} u_{\theta}^2 d\theta = 2\alpha^2 \int_{\Omega} u (u^{\frac{1}{2}})_{\theta\theta} d\theta,$$

and the third summand we rewrite as

$$-\int_{\Omega} u^{-\frac{1}{2}} u_{\theta} \sin \theta \, d\theta = 2 \int_{\Omega} u^{\frac{1}{2}} \cos \theta \, d\theta .$$

Inserting these identities into Eq. (3.3) and completing the square we arrive at

$$\begin{aligned} \frac{dS(u)}{dt} &= c_n \left\{ -\int_{\Omega} u^{\frac{1}{2}} \left( 2(u^{\frac{1}{2}})_{\theta\theta} - \frac{\alpha^2}{2} u^{\frac{1}{2}} \right)^2 d\theta + \frac{\alpha^4}{4} \int_{\Omega} u^{\frac{3}{2}} d\theta + 2 \int_{\Omega} u^{\frac{1}{2}} \cos \theta \, d\theta \right\} \\ &\leq c_n \left\{ \frac{\alpha^4}{4} M \sqrt{\|u\|_{\infty}} + 2\sqrt{M\pi} \right\} . \end{aligned} \quad (3.4)$$

The claim follows from Eq. (2.1) of Lemma 1, □

The entropy inequality in Eq. (3.2) holds also for many types of weak solutions of Eq. (1.1) that are obtained as limits of classical solutions of suitable regularizations. We demonstrate this for the strong generalized solutions in the case  $n = 3$  that were recently constructed by Chugunova, Pugh, and Taranets in [10, Theorem 2]. For  $\varepsilon > 0$ , consider the regularized equation

$$u_t + [f_{\varepsilon}(u) (u_{\theta\theta\theta} + \alpha^2 u_{\theta} - \sin \theta)]_{\theta} = 0, \quad \theta \in \Omega, \quad (3.5)$$

where  $f_{\varepsilon}(z) = \frac{z^4}{|z|+\varepsilon}$ . This equation has strictly positive classical solutions positive initial data in  $H^1(\Omega)$  (see [10, Lemma 3.4]), and the energy in Eq. (1.2) is dissipated.

The key step is to extend the entropy method to the regularized equation. Set

$$S_{\varepsilon}(u) = \int_{\Omega} s_{\varepsilon}(u) \, dx, \quad (3.6)$$

where  $s_{\varepsilon}(z) = z^{-\frac{3}{2}}(1 + \frac{3}{7}\varepsilon z^{-1})$  is chosen so that  $s_{\varepsilon}''(z)f_{\varepsilon}(z) = c_3 z^{-\frac{1}{2}}$ . Then  $\frac{dS_{\varepsilon}}{dt}$  satisfies the entropy identity in Eq. (3.4) along solutions of Eq. (3.5), and hence

$$S_{\varepsilon}(u(\cdot, t)) - S_{\varepsilon}(u(\cdot, 0)) \leq K_0 t,$$

with the same constant as in Lemma 4. As  $\varepsilon \rightarrow 0$  along a suitable subsequence, solutions of Eq. (3.5) converge uniformly to solutions of the original problem in Eq. (1.1), and the values of the energy and entropy also converge for all times, provided they are finite at  $t = 0$ . Thus the entropy inequality in Eq. (3.2) remains valid in the limit. This can be used to establish additional regularity properties: Using a suitable subsequence where  $u_{\theta\theta}$  converges weakly in  $L^2(\Omega \times (0, T))$  one can show that for every  $T > 0$ ,

$$\int_0^T \int_{\Omega} u^{\frac{1}{2}} \left( 2(u^{\frac{1}{2}})_{\theta\theta} - \frac{\alpha^2}{2} u^{\frac{1}{2}} \right)^2 d\theta dt \leq S(u(\cdot, 0)) + K_0 T < \infty,$$

and conclude that  $\int_{\Omega} u^{\frac{1}{2}} \left( (u^{\frac{1}{2}})_{\theta\theta} \right)^2 d\theta$  is finite for almost every  $t > 0$ .

Our final result concerns the dynamics near the energy minimizer, see Figure 4.

**Theorem 2 (Bounds on the speed of convergence.)** *Consider Eq. (1.1) with  $\omega = 0$ , and set  $\beta = n - \frac{3}{2}$ . Let  $u$  be a solution of mass  $M$  that dissipates energy and satisfies Eq. (3.2) with constants  $S_0$  and  $K_0$ , and let  $u^*$  be the energy-minimizing steady state of the same mass.*

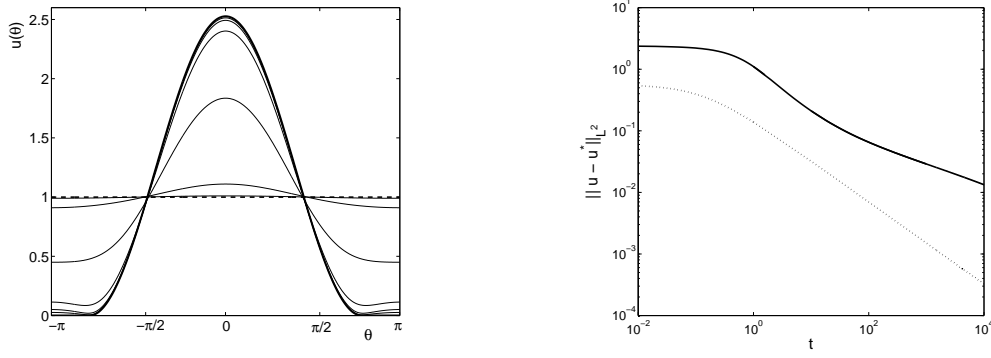


Figure 4: Evolution of a solution with  $\alpha = 1$ ,  $n = 3$  and initial data  $u_0 = 1$ . On the left: Time shots of the numerical solution at  $t = 0, 10^{-2}, 10^{-1}, 1, 10, 10^2, 10^3$ . On the right:  $L^2$ -distance of the solution from the energy minimizer. The dashed line shows the bound from Theorem 2.

- If  $u^*$  is strictly positive, set  $\mu = (1 - \alpha^2)(\min u^*)^n$  and  $\varepsilon_0 = \frac{(1 - \alpha^2)\pi}{2}(\min u^*)^2$ . For each solution with  $E(u(\cdot, 0)) < E(u^*) + \varepsilon_0$  there exists a constant  $K_1$  such that

$$\|u(\cdot, t) - u^*\|_{H^1} \leq K_1 e^{-\mu t};$$

- if  $u^*$  vanishes on an interval of positive length  $L$ , and  $n > \frac{3}{2}$ , then

$$\|u(\cdot, t) - u^*\|_2 \geq L^{1+\frac{\beta}{2}}(S_0 + K_0 t)^{-\frac{1}{\beta}};$$

- if  $u^*$  vanishes quadratically at a point, and  $n > \frac{5}{2}$ , then there exist positive constants  $K_2$  and  $K_3$  such that

$$\|u(\cdot, t) - u^*\|_2 \geq (K_2 + K_3 t)^{-\frac{3}{2(\beta-1)}}.$$

*Proof.* If  $u^*$  is strictly positive, then  $\alpha < 1$  by Theorem 1, and  $E$  is strictly convex. The Taylor expansion of  $E$  about  $u^*$  terminates after the quadratic term, because  $E$  itself is quadratic, and the linear term vanishes because  $u^*$  is a critical point where the positivity constraint is not active, and therefore

$$E(u) = E(u^*) + \frac{1}{2} \int_{\Omega} (u - u^*)_{\theta}^2 - \alpha^2 (u - u^*)^2 d\theta. \quad (3.7)$$

Since  $u$  and  $u^*$  have the same mass, the energy difference dominates the  $H^1$ -distance,

$$E(u) - E(u^*) \geq \frac{1 - \alpha^2}{2} \|(u - u^*)_{\theta}\|_2^2 \geq \frac{(1 - \alpha^2)\pi}{2} \|u - u^*\|_{\infty}^2.$$

Set  $\varepsilon = E(u(\cdot, 0)) - E(u^*) < \varepsilon_0$ , then  $\min u \geq (1 - \frac{\varepsilon}{\varepsilon_0}) \min u^* > 0$ . This means that  $u$  is a strictly positive, classical solution that can be differentiated as often as necessary. We compute the  $L^2$ -gradient of  $E$  from Eq. (3.7) as  $\frac{\delta E}{\delta u} = -(u - u^*)_{\theta\theta} - \alpha^2(u - u^*)$ . By Eq. (1.3) the energy is dissipated at rate

$$\frac{d}{dt} E(u(\cdot, t)) = - \int_{\Omega} u^n [(u - u^*)_{\theta\theta} + \alpha^2(u - u^*)]_{\theta}^2 d\theta$$

$$\begin{aligned}
&\leq -(\min u)^n \cdot \int_{\Omega} [(u - u^*)_{\theta\theta\theta} + \alpha^2(u - u^*)_{\theta}]^2 d\theta \\
&\leq -2 \left( \left(1 - \frac{\varepsilon}{\varepsilon_0}\right) \min u^* \right)^n (1 - \alpha^2) (E(u) - E(u^*)).
\end{aligned}$$

In the last step, we have used Parseval's identity to rewrite the integral and the energy difference in terms of the Fourier coefficients of  $u - u^*$ , and estimated the Fourier multipliers by

$$p^2(p^2 - \alpha^2)^2 \geq (1 - \alpha^2)(p^2 - \alpha^2), \quad (p \neq 0).$$

Exponential convergence of the energy follows from Gronwall's lemma. Since  $\min u$  converges exponentially to  $\min u^*$ , we conclude that  $E(u(\cdot, t)) - E(u^*) \leq K e^{-2\mu t}$  for some constant  $K$ , and the first claim follows.

If  $u^*$  vanishes on an interval of length  $L > 0$ , we apply Jensen's inequality to the convex function  $y \mapsto y^{-\frac{\beta}{2}}$  on this interval to obtain

$$S(u(\cdot, t)) \geq L^{1+\frac{\beta}{2}} \|u(\cdot, t) - u^*\|_2^{-\beta}.$$

The second claim follows by using the bound on the entropy in Eq. (3.2) and solving for the distance  $\|u(\cdot, t) - u^*\|_2$ .

If  $u^*$  vanishes quadratically at a point, we consider the interval of length  $L$  centered at that point and obtain with the same calculation as for the second case that

$$\begin{aligned}
\|u(\cdot, t) - u^*\|_2 &\geq \|u(\cdot, t)I_{|\theta| \geq \tau}\|_2 - \|u^*I_{|\theta| > \tau}\|_2 \\
&\geq L^{\frac{1}{\beta} + \frac{1}{2}} (S_0 + K_0 t)^{-\frac{1}{\beta}} - O(L^{\frac{3}{2}}).
\end{aligned}$$

The proof is completed by choosing  $L = \varepsilon(S_0 + K_0 t)^{-\frac{1}{\beta-1}}$  for  $\varepsilon > 0$  sufficiently small.  $\square$

Consider specifically the case of Eq. (1.1) where  $n = 3$ ,  $\alpha = 1$ , and  $\omega = 0$ . By Theorem 1, the energy minimizer vanishes on an interval of length  $L = 2(\pi - \tau) > 0$ , and by Theorem 2, solutions cannot converge to the minimizer more quickly than  $t^{-\frac{2}{3}}$ . We suspect that the distance from the steady state actually decays with  $t^{-\frac{1}{3}}$ .

This conjecture is supported by simulations, and by analogy with aggregation processes where the convergence to states is governed by power laws. In such processes, the speed of convergence is limited by the rate at which mass can be transferred from a region of low density to the region of accumulation. One example is the Lifshitz-Slyozov cubic law, which describes late-stage grain growth in alloys and the evaporation-condensation mechanism in supersaturated solutions [18]. A second example is the separation of a water drop from a tap, shown on the left of Figure 5. The drop slowly draws water from the tap through a thin neck. The right side of Figure 5 shows the accumulation of the mass of a solution of Eq. (1.1) at the bottom of the cylinder. The large drop at the bottom grows by slowly pulling mass from the small drop on the top through the thin bridges that connect them.

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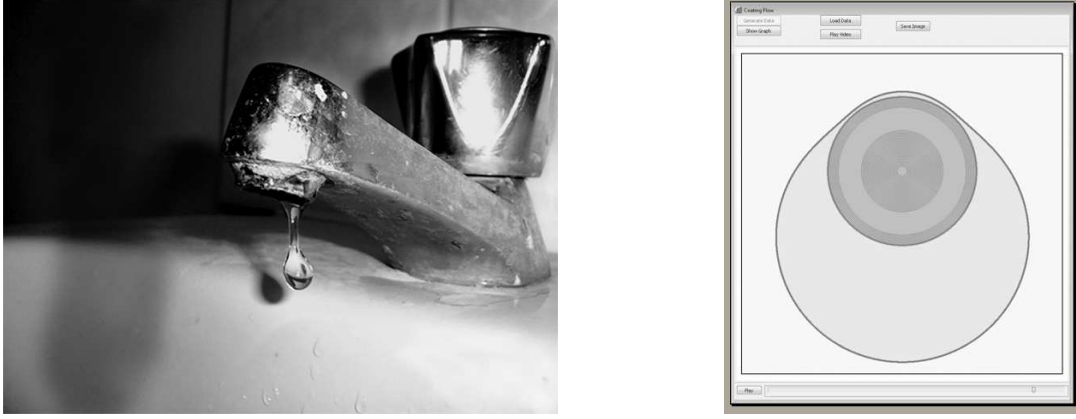


Figure 5: On the left: Separation of a water drop at a tap. Photograph “Perfect Back and White — Water drop”, by Elliott Minns [19]. On the right: Matlab simulation of the long-time behavior of a solution of Eq. (1.1) with parameters  $n = 3$ ,  $\alpha = 1$ , and  $\omega = 0$ . The thickness of the film is exaggerated to emphasize the shape.

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